Partial Solution Set, Leon §3.1

3.1.3 We are to show that the set $\mathbf{C}$ of complex numbers, with scalar multiplication defined by $\alpha(a+bi) = \alpha a + \alpha bi$ and addition defined by $(a+bi)+(c+di) = (a+c)+(b+d)i$, satisfies the eight axioms of a vector space. This is only a partial solution.

A1: Let $a+bi, c+di \in \mathbf{C}$. Then

$$\begin{aligned}(a+bi)+(c+di) &= ((a+c)+(b+d)i) && \text{(By definition of complex addition)}\\ &= ((c+a)+(d+b)i) && \text{(Real addition is commutative)}\\ &= (c+di)+(a+bi) && \text{(By definition of complex addition)}\end{aligned}$$

A2: Similar to A1; pick three complex numbers, use the definition of complex addition as often as necessary, together with the known associativity of real addition, to show that complex addition is associative.

A3: The zero element is $\mathbf{0} = (0+0i)$.

A4: To show existence of the additive inverse, choose an arbitrary complex number (say, $\mathbf{x} = a+bi$) and *construct* its additive inverse. This will be made easy by your knowledge of real additive inverses.

A5: We must prove that scalar multiplication distributes over complex addition. Let $a+bi, c+di \in \mathbf{C}$, and let $\alpha \in \mathbf{R}$. Then

$$\begin{aligned}\alpha((a+bi)+(c+di)) &= \alpha((a+c)+(b+d)i) && \text{(Def'n complex addition)}\\ &= \alpha(a+c)+\alpha(b+d)i && \text{(Def'n of scalar mult. in } \mathbf{C})\\ &= (\alpha a+\alpha c)+(\alpha b+\alpha d)i && \text{(Distributivity in } \mathbf{R})\\ &= (\alpha a+\alpha bi)+(\alpha c+\alpha di) && \text{(Def'n of complex addition)}\\ &= \alpha(a+bi)+\alpha(c+di) && \text{(Def'n of scalar mult. in } \mathbf{C})\end{aligned}$$

A6: Similar to A5.

A7: Use definition of scalar multiplication in $\mathbf{C}$ and associativity of real multiplication.

A8: Use definition of scalar multiplication in $\mathbf{C}$ and the fact that 1 is the multiplicative identity in $\mathbf{R}$.

3.1.4 Use the solution to 3.1.3 as a template for your solution. The objects are different (matrices rather than complex numbers) and the operations are necessarily defined differently, but these differences have no effect on the structure - $\mathbf{R}^{m\times n}$ is simply another vector space. The challenge is to avoid committing yourself to concrete values of m and/or n.

3.1.6 You can use either my solution to 3.1.3 or your own solution to 3.1.4 as a guide. If you found #4 easy, you might skip this one. If you found #4 difficult, then by all means do this one if you have time.

3.1.7 Show that the element $\mathbf{0}$ in a vector space is unique.

Note: This is a standard uniqueness argument. We assume that we have two zero elements and then discover that they are identical twins. The proof goes like this:

Proof: Let V be a vector space. We know that V contains at least one zero element, since V satisfies the axioms. We must show, then, that V contains *at most* one zero element. So suppose that $\mathbf{v}$ and $\mathbf{w}$ are zeros in V. Then

$$\begin{aligned} \mathbf{v} &= \mathbf{v}+\mathbf{w} && \text{(Since } \mathbf{w} \text{ is a zero)} \\ &= \mathbf{w}+\mathbf{v} && \text{(Since addition commutes)} \\ &= \mathbf{w} && \text{(Since } \mathbf{v} \text{ is a zero)} \end{aligned}$$

Thus uniqueness is proven, and it now makes sense to reserve a special symbol ($\mathbf{0}$) to denote the zero element. □

3.1.11 Let V be the set of all ordered pairs of real numbers with addition defined in the usual fashion by $(x_1, x_2)+(y_1, y_2) = (x_1+y_1, x_2+y_2)$, but with scalar multiplication defined by $\alpha \circ (x_1, x_2) = (\alpha x_1, x_2)$. Is V a vector space with these operations? Justify your answer.

Solution: No, this is not a vector space. Axiom 6 fails.

3.1.12 Let $\mathbf{R}^+$ denote the set of positive real numbers. Define the operation of scalar multiplication, denoted $\circ$, by $\alpha \circ x = x^\alpha$ for any real α and $x \in \mathbf{R}^+$. Define addition, denoted $\oplus$, by $x \oplus y = x \cdot y$ for all $x, y \in \mathbf{R}^+$. (The dot represents the usual multiplication of reals.) Is $\mathbf{R}^+$ a vector space when equipped with these operations? Prove your answer.

Solution: Yes, this is a vector space. To prove this, we must verify that the axioms hold. Here is a partial proof:

A1: Use the definition of $\oplus$, together with the commutativity of ordinary real multiplication.

A2: Use the definition of $\oplus$, together with the associativity of ordinary real multiplication.

A3: The zero element is the number 1, since for any $x \in \mathbf{R}^+$ we have $x \oplus 1 = x \cdot 1 = x$.

A4: The additive inverse in this oddball space is the usual *multiplicative* inverse. That is, for any $x \in \mathbf{R}^+$, $1/x \in \mathbf{R}^+$, and $x \oplus 1/x = x \cdot 1/x = 1$. By the preceding argument, 1 is the zero element.

A5: Let $\alpha \in \mathbf{R}$ and $x, y \in \mathbf{R}^+$. Then

$$\begin{aligned} \alpha \circ (x \oplus y) &= \alpha \circ (x \cdot y) \\ &= (x \cdot y)^\alpha \\ &= x^\alpha \cdot y^\alpha \\ &= (\alpha \circ x) \cdot (\alpha \circ y) \\ &= (\alpha \circ x) \oplus (\alpha \circ y) \end{aligned}$$

A6: Let $\alpha, \beta, x \in \mathbf{R}$. Then

$$\begin{aligned}(\alpha+\beta)x &= x^{\alpha+\beta} \\ &= x^{\alpha}x^{\beta} \\ &= \alpha x \oplus \beta x\end{aligned}$$

A7: Let $\alpha, \beta \in \mathbf{R}$, and $x \in \mathbf{R}^{+}$. Then

$$\begin{aligned}(\alpha\beta) \circ x &= x^{\alpha\beta} \\ &= x^{\beta\alpha} \\ &= \left(x^{\beta}\right)^{\alpha} \\ &= \alpha \circ \left(x^{\beta}\right) \\ &= \alpha \circ (\beta \circ x)\end{aligned}$$

A8: Let $x \in \mathbf{R}$. Then $1 \cdot x = x^1 = x$, where the first equality is by our local definition of scalar multiplication and the second is by the usual laws of exponents.